X. *On the Motion of Bodies affected by Friction. By the Rev. Samuel Vince, A. M. of Cambridge; communicated by Anthony Shepherd, D. D. F. R. S. Plumian Professor of Astronomy and experimental Philosophy at Cambridge.*

Read November 25, 1784.

THE subject of the paper which I have now the honour of presenting to the Royal Society, seems to be of a very considerable importance both to the practical mechanic and to the speculative philosopher; to the former, as a knowledge of the laws and quantity of the friction of bodies in motion upon each other will enable him at first to render his machines more perfect, and save him in a great measure the trouble of correcting them by trials; and to the latter, as those laws will furnish him with principles for his theory, which when established by experiments will render his conclusions applicable to the real motion of bodies upon each other. But, however important a part of mechanics this subject may constitute, and however, from its obvious uses, it might have been expected to have claimed a very considerable attention both from the mechanic and philosopher, yet it has, of all the other parts of this branch of natural philosophy, been the most neglected. The law by which the motions of bodies are retarded by friction has never, that I know of, been truly established, MUSSCHENBROEK says, that in small velocities the friction varies very nearly as the velocity, but that in great velocities the friction increases; he has also attempted to prove, that by increasing

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the weight of a body the friction does not always increafe exactly in the fame ratio; and that the fame body, if by changing its pofition you change the magnitude of the furface on which it moves, will have its quantity of friction alfo changed. Hel-sham and Ferguson, from the fame kind of experiments, have endeavoured to prove, that the friction does *not* vary by changing the quantity of furface on which the body moves; and the latter of thefe afferts, that the friction increafes very nearly as the velocity; and that by increafing the weight, the friction is increafed in the *fame* ratio. Thefe different con-clufions induced me to repeat their experiments, in order to fee how far they were conclufive in refpect to the principles de-duced from them: when it appeared, that there was another caufe operating befides friction, which they had not attended to, and which rendered all their deductions totally inconclufive. Of thofe who have written on the theory, no one has efta-blifhed it altogether on true principles: Euler (whofe theory is extremely elegant, and which, as he has fo fully confidered the fubject, would have precluded the neceffity of offering any thing further, had its principles been founded on experiments) fuppofes the friction to vary in proportion to the velocity of the body, and its preffure upon the plane, neither of which are true: and others, who have imagined that friction is a uni-formly retarding force (and which conjecture will be confirmed by our experiments), have ftill retained the other fuppofition, and therefore rendered their folutions not at all applicable to the cafes for which they were intended. I therefore endea-voured by a fet of experiments to determine,

　1ft, *Whether friction be a uniformly retarding force,*

　2dly, *The quantity of friction.*

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3dly, *Whether the friction varies in proportion to the preſſure or weight.*

4thly, *Whether the friction be the ſame on whichever of its ſur-faces a body moves.*

The experiments, in which I was aſſiſted by my ingenious friend the Rev. Mr. JONES, Fellow of Trinity College, were made with the utmoſt care and attention, and the ſeveral reſults agreed ſo very exactly with each other, that I do not ſcruple to pronounce them to be concluſive.

2. A plane was adjuſted parallel to the horizon, at the extremity of which was placed a pulley, which could be elevated or depreſſed in order to render the ſtring which connected the body and the moving force parallel to the plane. A ſcale accurately divided was placed by the ſide of the pulley perpendicular to the horizon, by the ſide of which the moving force deſcended; upon the ſcale was placed a moveable ſtage, which could be adjuſted to the ſpace through which the moving force deſcended in any given time, which time was meaſured by a well regulated pendulum clock vibrating ſeconds. Every thing being thus prepared, the following experiments were made to aſcertain the law of friction. But let me firſt obſerve, that if friction be a uniform force, the difference between it and the given force of the moving power muſt be alſo uniform, and therefore the moving body muſt deſcend with a uniformly accelerated velocity, and conſequently the ſpaces deſcribed from the beginning of the motion muſt be as the ſquares of the times, juſt as when there was no friction, only they will be diminiſhed on account of the friction.

3. EXP. I. A body was placed upon the horizontal plane, and a moving force applied, which from repeated trials was found to deſcend $52\frac{1}{2}$ inches in $4''$, for by the beat of the clock and

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the found of the moving force when it arrived at the ftage, the fpace could be very accurately adjufted to the time; the ftage was then removed to that point to which the moving force would defcend in 3″, upon fuppofition that the fpaces defcribed by the moving power were as the fquares of the times; and the fpace was found to agree very accurately with the time; the ftage was then removed to that point to which the moving force ought to defcend in 2″, upon the fame fuppofition, and the defcent was found to agree exactly with the time; laftly, the ftage was adjufted to that point to which the moving force ought to defcend in 1″, upon the fame fuppofition, and the fpace was obferved to agree with the time. Now, in order to find whether a difference in the time of defcent could be obferved, by removing the ftage a little above and below the pofitions which correfponded to the above times, the experiment was tried, and the defcent was always found too foon in the former, and too late in the latter cafe; by which I was affured that the fpaces firft mentioned correfponded exactly to the times. And, for the greater certainty, each defcent was repeated eight or ten times; and every caution ufed in this experiment was alfo made ufe of in all the following.

Exp. 2. A fecond body was laid upon the horizontal plane, and a moving force applied which defcended 41¾ inches in 3″; the ftage was then adjufted to the fpace correfponding to 2″, upon fuppofition that the fpaces defcended through were as the fquares of the times, and it was found to agree accurately with the time; the ftage was then adjufted to the fpace correfponding to 1″, upon the fame fuppofition, and it was found to agree with the time.

Exp. 3. A third body was laid upon the horizontal plane, and a moving force applied, which defcended 59¾ inches in 4″; the

ftage was then adjufted to the fpace correfponding to 3″, upon fuppofition that the fpaces defcended through were as the fquares of the times, and it was found to agree with the time; the ftage was then adjufted to the fpace correfponding to 2″, upon the fame fuppofition, and it was found to agree with the time; the ftage was then adjufted to the fpace correfponding to 1″, and was found to agree with the time.

Exp. 4. A fourth body was then taken and laid upon the horizontal plane, and a moving force applied, which defcended 55 inches in 4″; the ftage was then adjufted to the fpace through which it ought to defcend in 3″, upon fuppofition that the fpaces defcended through were as the fquares of the times, and it was found to agree with the time; the ftage was then adjufted to the fpace correfponding to 2″, upon the fame fuppofition, and was found to agree with the time; laftly, the ftage was adjufted to the fpace correfponding to 1″, and it was found to agree exactly with the time.

Befides thefe experiments, a great number of others were made with hard bodies, or thofe whofe parts fo firmly cohered as not to be moved *inter fe* by the friction; and in each experiment bodies of very different degrees of friction were chofen, and the refults all agreed with thofe related above; we may therefore conclude, that *the friction of hard bodies in motion is a uniformly retarding force.*

But to determine whether the fame was true for bodies when covered with cloth, woollen, &c. experiments were made in order to afcertain it; when it was found in all cafes, that the retarding force increafed with the velocity; but, upon covering bodies with paper, the confequences were found to agree with thofe related above.

4. Having

4. Having proved that the retarding force of all hard bodies arising from friction is uniform, the quantity of friction, confidered as equivalent to a weight without inertia drawing the body on the horizontal plane backwards, or acting contrary to the moving force, may be immediately deduced from the foregoing experiments. For let $M =$ the moving force expreffed by its weight; $F =$ the friction; $W =$ the weight of the body upon the horizontal plane; $S =$ the fpace through which the moving force defcended in the time t expreffed in feconds; $r = 16\frac{1}{12}$ feet; then the whole accelerative force (the force of gravity being unity) will be $\frac{M-F}{M+W}$; hence, by the laws of uniformly accelerated motions, $\frac{M-F}{M+W} \times r t^2 = S$, confequently $F = M - \frac{\overline{M+W} \times S}{r t^2}$. To exemplify this, let us take the cafe of the laft experiment, where $M = 7$, $W = 25\frac{3}{4}$, $S = 4\frac{7}{12}$ feet, $t = 4''$; hence $F = 7 - \frac{32\frac{3}{4} \times 4\frac{7}{12}}{16\frac{1}{12} \times 16} = 6.417$; confequently the friction was to the weight of the rubbing body as 6.4167 to 25.75. And the great accuracy of determining the friction by this method is manifeft from hence, that if an error of 1 inch had been made in the defcent (and experiments carefully made may always determine the fpace to a much greater exactnefs) it would not have affected the conclufion $\frac{1}{600}$dth part of the whole.

5. We come in the next place to determine, whether friction, *cæteris paribus*, varies in proportion to the weight or preffure. Now if the whole quantity of the friction of a body, meafured by a weight without inertia equivalent to the friction drawing the body backwards, increafes in proportion to its weight, it is manifeft, that the retardation of the velocity of the body arifing from the friction will not be altered; for the

                        retardation

retardation varies as $\frac{\text{Quantity of friction}}{\text{Quantity of matter}}$; hence, if a body be put in motion upon the horizontal plane by any moving force, if both the weight of the body and the moving force be increafed in the fame ratio, the acceleration arifing from that moving force will remain the fame, becaufe the accelerative force varies as the moving force divided by the whole quantity of matter, and both are increafed in the fame ratio; and if the quantity of friction increafes alfo as the weight, then the retardation arifing from the friction will, from what has been faid, remain the fame, and therefore the whole acceleration of the body will not be altered; confequently the body ought, upon this fuppofition, ftill to defcribe the fame fpace in the fame time. Hence, by obferving the fpaces defcribed in the fame time, when both the body and the moving force are increafed in the fame ratio, we may determine whether the friction increafes in proportion to the weight. The following experiments were therefore made in order to afcertain this matter.

Exp. 1. A body weighing 10 oz. by a moving force of 4 oz. defcribed in 2″ a fpace of 51 inches; by loading the body with 10 oz. and the moving force with 4 oz. it defcribed 56 inches in 2″; and by loading the body again with 10 oz. and the moving force with 4 oz. it defcribed 63 inches in 2″.

Exp. 2. A body, whofe weight was 16 oz. by a moving force of 5 oz. defcribed a fpace of 49 inches in 3″; and by loading the body with 64 oz. and the moving force with 20 oz. the fpace defcribed in the fame time was 64 inches.

Exp. 3. A body weighing 6 oz. by a moving force of 2½ oz. defcribed 28 inches in 2″; and by loading the body with 24 oz. and the moving force with 10 oz. the fpace defcribed in the fame time was 54 inches.

Exp.

E<small>XP</small>. 4. A body weighing 8 oz. by a moving force of 4 oz. deſcribed 33½ inches in 2″; and by loading the body with 8 oz. and the moving force with 4 oz. the ſpace deſcribed in the ſame time was 47 inches.

E<small>XP</small>. 5. A body whoſe weight was 9 oz. by a moving force of 4½ oz. deſcribed 48 inches in 2″; and by loading the body with 9 oz. and the moving force with 4½ oz. the ſpace deſcribed in the ſame time was 60 inches.

E<small>XP</small>. 6. A body weighing 10 oz. by a moving force of 3 oz. deſcribed 20 inches in 2″; by loading the body with 10 oz. and the moving force with 3 oz. the ſpace deſcribed in the ſame time was 31 inches; and by loading the body again with 30 oz. and the moving force with 9 oz. the ſpace deſcribed was 34 inches in 2″.

From theſe experiments, and many others which it is not neceſſary here to relate, it appears, that the ſpace deſcribed is always increaſed by increaſing the weight of the body and the accelerative force in the ſame ratio; and as the acceleration ariſing from the moving force continued the ſame, it is manifeſt, that the retardation ariſing from the friction muſt have been diminiſhed, for the whole accelerative force muſt have been increaſed on account of the increaſe of the ſpace deſcribed in the ſame time; and hence (as the retardation from friction varies as $\frac{\text{Quantity of friction}}{\text{Quantity of matter}}$) *the quantity of friction increaſes in a leſs ratio than the quantity of matter or weight of the body.*

6. We come now to the laſt thing which it was propoſed to determine, that is, whether the friction varies by varying the ſurface on which the body moves. Let us call two of the ſurfaces A and *a*, the former being the greater, and the latter the leſs. Now the weight on every given part of *a* is as much greater

than the weight on an equal part of A, as A is greater than *a* ; if therefore the friction was in proportion to the weight, *cæteris paribus*, it is manifest, that the friction on *a* would be equal to the friction on A, the whole friction being, upon such a suppofition, as the weight on any given part of each surface multiplied into the number of such parts or into the whole area, which products, from the proportion above, are equal. But from the laft experiments it has been proved, that the friction on any given surface increafes in a lefs ratio than the weight; confequently the friction on any given part of *a* has a lefs ratio to the friction on an equal part of A than A has to *a*, and hence the friction on *a* is lefs than the friction on A, that is, the fmalleft surface has always the leaft friction. But as this conclufion is contrary to the generally received opinion, I have thought it proper to confirm the fame by a fet of experiments. But before I proceed to relate them, I will beg leave to recommend to thofe, who may afterwards be induced to repeat them, the following cautions, which are extremely neceffary to be attended to. Great care muft be taken that the two furfaces have exactly the fame degree of roughnefs ; in order to be certain of which, fuch bodies muft be chofen as have no knots in them, and whofe grain is fo very regular that when the two furfaces are planed with a fine rough plane, their roughnefs may be the fame, which will not be the cafe if the body be knotty, or the grain irregular, or if it happens not to run in the fame direction on both furfaces. When you cannot depend on the furfaces having the fame degree of roughnefs, the beft way will be to pafte fome fine rough paper on each furface, which perhaps will give a more equal degree of roughnefs than can be obtained by any other method. Now as the proof which I have already given depends only on the motion

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of the body upon the *fame* furface, it is not liable to any inaccuracy of the kind which the preceding cautions have been given to avoid, nor indeed to any other, and therefore it muft be perfectly conclufive. In the following experiments the cautions mentioned above were carefully attended to.

Exp. 1. A body was taken whofe flat furface was to its edge as 22 : 9, and with the fame moving force the body defcribed on its flat fide $33\frac{1}{2}$ inches in $2''$, and on its edge 47 inches in the fame time.

Exp. 2. A fecond body was taken whofe flat furface was to its edge as 32 : 3, and with the fame moving force it defcribed on its flat fide 32 inches in $2''$, and on its edge it defcribed $37\frac{1}{2}$ inches in the fame time.

Exp. 3. I took another body and covered one of its furfaces, whofe length was 9 inches, with a fine rough paper, and by applying a moving force, it defcribed 25 inches in $2''$; I then took off fome paper from the middle, leaving only $\frac{3}{8}$ of an inch at the two ends, and with the fame moving force it defcribed 40 inches in the fame time.

Exp. 4. Another body was taken which had one of its furfaces, whofe length was 9 inches, covered with a fine rough paper, and by applying a moving force it defcribed 42 inches in $2''$; fome of the paper was then taken off from the middle, leaving only $1\frac{3}{4}$ inches at the two ends, and with the fame moving force it defcribed 54 inches in $2''$; I then took off more paper, leaving only $\frac{1}{4}$ of an inch at the two ends, and the body then defcribed, by the fame moving force, 60 inches in the fame time.

In the two laft experiments the paper which was taken off the furface was laid on the body, that its weight might not be altered.

Exp.

EXP. 5. A body was taken whofe flat furface was to its edge as 30 : 17; the *flat* fide was laid upon the horizontal plane, a moving force was applied, and the ftage was fixed in order to ftop the moving force, in confequence of which the body would then go on with the velocity acquired until the friction had deftroyed all its motion; when it appeared from a mean of 12 trials that the body moved, after its acceleration ceafed, $5\frac{2}{5}$ inches before it ftopped. The *edge* was then applied, and the moving force defcended through the fame fpace, and it was found, from a mean of the fame number of trials, that the fpace defcribed was $7\frac{1}{3}$ inches before the body loft all its motion, after it ceafed to be accelerated.

EXP. 6. Another body was then taken whofe flat furface was to its edge as 60 : 19, and, by proceeding as before, on the flat furface it defcribed, at a mean of 12 trials, $5\frac{1}{4}$ inches, and on the edge $6\frac{1}{24}$ inches, before it ftopped, after the acceleration ceafed.

EXP. 7. Another body was taken whofe flat furface was to its edge as 26 : 3, and the fpaces defcribed on thefe two fur-faces, after the acceleration ended, were, at a mean of 10 trials, $4\frac{3}{7}$ and $7\frac{7}{10}$ inches refpectively.

From all thefe different experiments it appears, that the fmalleft furface had always the leaft friction, which agrees with the confequence deduced from the confideration that the friction does not increafe in fo great a ratio as the weight; we may therefore conclude, that *the friction of a body does not con-tinue the fame when it has different furfaces applied to the plane on which it moves, but that the fmalleft furface will have the leaft friction.*

7. Having thus eftablifhed, from the moft decifive experi-ments, all that I propofed relative to friction, I think it proper, before

before I conclude, to give the refult of my examination into the nature of the experiments which have been made by others; which were repeated, in order to fee how far they were conclufive in refpect to the principles which have been deduced from them. The experiments which have been made by all the authors that I have feen, have been thus inftituted. To find what moving force would *juft* put a body at reft in motion : and they concluded from thence, that the accelerative force was then equal to the friction; but it is manifeft, that any force which will put a body in motion muft be *greater* than the force which oppofes its motion, otherwife it could not overcome it; and hence, if there were no other objection than this, it is evident, that the friction could not be very accurately obtained; but there is another objection which totally deftroys the experiment fo far as it tends to fhow the quantity of friction, which is the ftrong cohefion of the body to the plane when it lies at reft; and this is confirmed by the following experiments. 1ft, A body of 12¾ oz. was laid upon an horizontal plane, and then loaded with a weight of 8 lb. and fuch a moving force was applied as would, when the body was juft *put* in motion, continue that motion without any acceleration, in which cafe the friction muft be juft equal to the accelerative force. The body was then ftopped, when it appeared, that the fame moving force which had *kept* the body in motion before, would not *put* it in motion, and it was found neceffary to take off 4½ oz. from the body before the fame moving force *would* put it in motion; it appears, therefore, that this body, when laid upon the plane at reft, acquired a very ftrong cohefion to it. 2dly, A body whofe weight was 16 oz. was laid at reft upon the horizontal plane, and it was found that a moving force of 6 oz. would juft *put* it in motion; but that a moving force of 4 oz. *would*

would, when it was juſt put in motion, *continue* that motion without any acceleration, and therefore the accelerative force muſt *then* have been equal to the friction, and not when the moving force of 6 oz. was applied.

From theſe experiments therefore it appears, how very conſiderable the coheſion was in proportion to the friction when the body was in motion ; it being, in the latter caſe, almoſt $\frac{1}{3}$, and in the former it was found to be very nearly equal to the whole friction. All the concluſions therefore deduced from the experiments, which have been inſtituted to determine the friction from the force neceſſary to *put* a body in motion (and I have never ſeen any deſcribed but upon ſuch a principle) have manifeſtly been totally falſe ; as ſuch experiments only ſhew the reſiſtance which ariſes from the coheſion and friction conjointly.

8. I ſhall conclude this part of the ſubject with a remark upon Art. 5. It appears from all the experiments which I have made, that the proportion of the increaſe of the friction to the increaſe of the weight was different in all the different bodies which were made uſe of ; no general rule therefore can be eſtabliſhed to determine this for *all* bodies, and the experiments which I have hitherto made have not been ſufficient to determine it for the *ſame* body. At ſome future opportunity, when I have more leiſure, I intend to repeat the experiments in order to eſtabliſh, in ſome particular caſes, the law by which the quantity of friction increaſes by increaſing the weight. Leaving this ſubject therefore for the preſent, I ſhall proceed to eſtabliſh a theory upon the principles which we have already deduced from our experiments.

PROPOSITION I.

Let e, f, g, (fig. 1.) *reprefent either a cylinder, or that circular fection of a body on which it rolls down the inclined plane* CA *in confequence of its friction, to find the time of defcent and the number of revolutions.*

As it has been proved in Art. 5. that the friction of a body does not increafe in proportion to its weight or preffure, we cannot therefore, by knowing the friction on any other plane, determine the friction on CA ; the friction therefore on CA can only be determined by experiments made upon *that* plane, that is, by letting the body defcend from reft, and obferving the fpace defcribed in the firft fecond of time ; call that fpace *a*, and then, as by Art. 3. friction is a uniformly retarding force, the body muft be uniformly accelerated, and confequently the whole time of defcent in feconds will be $= \sqrt{\dfrac{AC}{a}}$. Now to determine the number of revolutions, let *s* be the center of ofcillation to the point of fufpenfion *a* * ; then, becaufe no force acting at *a* can affect the motion of the point *s*, that point, notwithftanding the action of the friction at *a*, will always have a motion parallel to CA uniformly accelerated by a force equal to that with which the body would be accelerated if it had no friction ; hence, if $2m = 32\frac{1}{6}$ feet, the velocity acquired by the point *s* in the firft fecond will be $= \dfrac{2m \times CB}{CA}$; now the excefs of the ve-

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* *a* and *s* are not fixed points in the body, but the former always reprefents that point of the body in contact with the plane, and the latter the correfponding center of ofcillation.

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locity of the point s above that of r (r being the center) is manifeſtly the velocity with which s is carried about r; hence the velocity of s about the center $= \frac{2m \times CB}{CA} - 2a = \frac{2m \times CB - 2a \times CA}{CA}$,

confequently $rs : ra :: \frac{2m \times CB - 2a \times CA}{CA} : \frac{2m \times ra \times CB - 2a \times ra \times CA}{rs \times CA}$

$=$ the velocity with which a point of the circumference is carried about the center, and which therefore expreſſes the force which accelerates the rotation; now as $2a$ expreſſes the accelerative force of the body down the plane, and the ſpaces deſcribed in the ſame time are in proportion to thoſe forces, we have $2a : CA :: \frac{2m \times ra \times CB - 2a \times ra \times CA}{rs \times CA} : \frac{m \times ra \times CB - a \times ra \times CA}{a \times rs}$
the ſpace which any point of the circumference deſcribes about the center in the whole time of the body's deſcent down CA; which being divided by the circumference $p \times ra$ (where $p = 6.283$ &c.) will give $\frac{m \times BC - a \times AC}{p \times a \times rs}$ for the whole number of revolutions required.

Cor. 1. If $a \times CA = m \times BC$, the number of revolutions $= 0$, and therefore the body will then only ſlide; confequently the friction vaniſhes.

Cor. 2. Let $a'r's'$ (fig. 2.) be the next poſition of ars, and draw $tr'b$ parallel to sa, then will $s't$ repreſent the retardation of the center r ariſing from friction, and $a'b$ will repreſent the acceleration of a point of the circumference about its center; hence the retardation of the center : acceleration of the circumference about the center :: $s't : a'b ::$ (by ſim. $\triangle$'s) $tr' : br' :: rs : ra$.

Cor. 3. If a' coincides with a, the body does not *ſlide* but only *roll*; now in this caſe $ss' : rr' :: as : ar$; but as ss' and rr' repreſent the ratio of the velocities of the points s and r,

they

they will be to each other as $\frac{2m \times \text{BC}}{\text{CA}}$: $2a$ or as $m \times \text{CB} : a \times \text{CA}$; hence, when the body *rolls* without *fliding*, $as : ar :: m \times \text{CB} : a \times \text{CA}$.

Cor. 4. The time of defcent down CA is $= \sqrt{\frac{\text{AC}}{a}}$; but by the laſt Cor. when the body *rolls* without *fliding*, $a = \frac{m \times ra \times \text{BC}}{sa \times \text{AC}}$, hence the time of defcent in that cafe $= \text{AC} \sqrt{\frac{sa}{m \times ra \times \text{BC}}}$; now the time of defcent, if there were no friction, would be $= \frac{\text{AC}}{\sqrt{m \times \text{BC}}}$, hence the time of defcent, when the body *rolls* without *fliding* : time of free defcent $:: \sqrt{sa} : \sqrt{ra}$.

Cor. 5. By the laſt Cor. it appears, that when the body juſt *rolls* without *fliding*, or when the friction is juſt equal to the accelerative force, the time of defcent $= \text{AC} \sqrt{\frac{sa}{m \times ra \times \text{BC}}}$; now it is manifeſt, that the time of defcent will continue the fame, if the friction be increafed, for the body will ſtill freely roll, as no increafe of the friction acting at a can affect the motion of the point s.

If the body be projected from C with a velocity, and at the fame time have a rotatory motion, the time of defcent and the number of revolutions may be determined from the common principles of uniformly accelerated motions, as we have already inveſtigated the accelerative force of the body down the plane and of its rotation about its axis; it feems therefore unneceſſary to lengthen out this paper with the inveſtigations.

PROPOSITION II.

Let the body be projected on an horizontal plane LM (fig. 3.) *with a given velocity, to determine the space through which the body will move before it stops, or before its motion becomes uniform.*

CASE I.   1. Suppose the body to have no rotatory motion when it begins to move; and let $a =$ the velocity of projection *per* second meafured in feet, and let the retarding force of the friction of the body, meafured by the velocity of the body which it can deftroy in one fecond of time, be determined by experiment and called F, and let x be the fpace through which the body would move by the time its motion was all deftroyed when projected with the velocity a, and retarded by a force F; then, from the principles of uniformly retarded motion, $x = \frac{a^2}{2F}$, and if $t =$ time of defcribing that fpace, we have $t = \frac{a}{F}$, and hence the fpace defcribed in the firft fecond of time $= \frac{2a - F}{2}$. Now it is manifeft, that when the rotatory motion of the body about its axis is equal to its progreffive motion, the point a will be carried backwards by the *former* motion as much as it is carried forwards by the *latter*; confequently the point of contact of the body with the plane will then have no motion in the direction of the plane, and hence the friction will at that inftant ceafe, and the body will continue to *roll* on uniformly without *fliding* with the velocity which it has at that point.   Put therefore $z =$ the fpace defcribed from the commencement of the motion till it becomes uniform, then the body being uniformly retarded, the fpaces from the end of

the

the motion vary as the fquares of the velocities, hence $\frac{a^2}{2F} : a^2 \left(:: 1 : 2F \right) :: \frac{a^2}{2F} - z : a^2 - 2Fz =$ fquare of the progref-five velocity when the motion becomes uniform; therefore the velocity deftroyed by friction $= a - \sqrt{a^2 - 2Fz}$; hence, as the velocity generated or deftroyed in the fame time is in proportion to the force, we have by Cor. 2. Prop. 1.

$r s : r a :: a - \sqrt{a^2 - 2Fz} : \frac{ra}{rs} \times \overline{a - \sqrt{a^2 - 2Fz}}$ the velocity of the circumference efg generated about the center, confequently $\sqrt{a^2 - 2Fz} = \frac{ra}{rs} \times \overline{a - \sqrt{a^2 - 2Fz}}$, and hence $z = \frac{rs^2 + 2rs \times ra \times a^2}{as^2 \times 2F}$ the fpace which the body defcribes before the motion becomes uniform.

2. If we fubftitute this value of z into the expreffion for the velocity, we fhall have $a \times \frac{ra}{rs}$ for the velocity of the body when its motion becomes uniform; hence therefore it appears, that the velocity of the body, when the friction ceafes, will be the fame whatever be the quantity of the friction. If the body be the circumference of a circle, it will always lofe half the ve-locity before its motion becomes uniform.

CASE II. 1. Let the body, befides having a progreffive velocity in the direction LM (fig. 3.) have alfo a rotatory mo-tion about its center in the direction gfe, and let v reprefent the initial velocity of any point of the circumference about the center, and fuppofe it firft to be lefs than a; then friction being a uniformly retarding force, no alteration of the velocity of the point of contact of the body upon the plane can affect the quantity of friction; hence the progreffive velocity of the body will be the fame as before, and confequently the rotatory velo-city

city generated by friction will also be the same, to which if we add the velocity about the center at the beginning of the motion, we shall have the whole rotatory motion; hence therefore, $v + \frac{ra}{rs} \times \overline{a - \sqrt{a^2 - 2Fz}} = \sqrt{a^2 - 2Fz}$, consequently $z = \frac{a^2 \times as^2 - \overline{v \times rs + a \times ra}^2}{2F \times as^2}$ the space described before the motion becomes uniform.

2. If this value of z be substituted into the expression for the velocity, we shall have $\frac{v \times rs + a \times ra}{as}$ for the velocity when the friction ceases.

3. If $v = a$, then $z = o$, and hence the body will continue to move uniformly with the first velocity.

4. If v be greater than a, then the rotatory motion of the point a on the plane being greater than its progressive motion and in a contrary direction, the absolute motion of the point a upon the plane will be in the direction ML, and consequently friction will now act in the direction LM in which the body moves, and therefore will accelerate the *progressive* and retard the *rotatory* motion; hence it appears, that *the progressive motion of a body may be* ACCELERATED *by friction.* Now to determine the space described before the motion becomes uniform, we may observe, that as the progressive motion of the body is now accelerated, the velocity after it has described any space z will be $= \sqrt{a^2 + 2Fz}$, hence the velocity acquired $= \sqrt{a^2 + 2Fz} - a$, and consequently the rotatory velocity destroyed $\frac{ra}{rs} \times \sqrt{a^2 + 2Fz} - a$, hence $v - \frac{ra}{rs} \times \sqrt{a^2 + 2Fz} - a = \sqrt{a^2 + 2Fz}$, therefore $z = \frac{\overline{rs \times v + ra \times a}^2 - a^2 \times as^2}{2F \times as^2}$ the space required.

5. If

5. If $a = o$, or the body be placed upon the plane without any progreffive velocity, then $z = \frac{rs^2 \times v^2}{2F \times as^2}$.

CASE III. 1. Let the given rotatory motion be in the direction $g\,e f$; then as the friction muft in this cafe always act in the direction ML, it muft continually tend to deftroy both the progreffive and rotatory motion. Now as the velocity deftroyed in the fame time is in proportion to the retarding force, and the force which retards the *rotatory* is to the force which retards the *progreffive* velocity by Cor. 2. Prop. 1. as $r\,a : r\,s$, therefore if v be to a as $r\,a$ is to $r\,s$, then the retarding forces being in proportion to the velocities, both motions will be deftroyed together, and confequently the body, after defcribing a certain fpace, will reft; which fpace, being that defcribed by the body uniformly retarded by the force F, will, from what was proved in Cafe I. be equal to $\frac{a^2}{2F}$.

2. If v bears a greater proportion to a than $r\,a$ does to $r\,s$, it is manifeft, that the rotatory motion will not be all deftroyed when the progreffive is; confequently the body, after it has defcribed the fpace $\frac{a^2}{2F}$, will return back in the direction ML; for the progreffive motion being then deftroyed, and the rotatory motion ftill continuing in the direction $g\,e f$, will caufe the body to return with an accelerative velocity until the friction ceafes by the body's beginning to roll, after which it will move on uniformly. Now to determine the fpace defcribed before this happens, we have $r\,s : r\,a :: a : \frac{r a \times a}{rs}$ the rotatory velocity deftroyed when the progreffive is all loft; hence $v - \frac{ra \times a}{rs} = \frac{v \times rs - a \times ra}{rs} =$ the rotatory velocity at that time, which

being

being fubftituted for v in the laft article of Cafe II. gives $\dfrac{\overline{v+rs-a\times ra}^2}{2F\times as^2}$ for the fpace defcribed before the motion becomes uniform.

3. If v has a lefs proportion to a than $r\,a$ has to $r\,s$, it is manifeft, that the *rotatory* motion will be deftroyed before the *progreffive*; in which cafe a rotatory motion will be generated in a contrary direction until the two motions become equal, when the friction will inftantly ceafe, and the body will then move on uniformly. Now $ra : rs :: v : \dfrac{v\times rs}{ra}$ the progreffive velocity deftroyed when the rotatory velocity ceafes, hence $a - \dfrac{v\times rs}{ra} = \dfrac{a\times ra - v\times rs}{ra} =$ progreffive velocity when it begins its rotatory motion in a contrary direction; fubftitute therefore this quantity for a in the expreffion for z in Cafe I. and we have $\dfrac{\overline{rs^2 + 2rs\times ra \times \overline{a\times ra - v\times rs}}^2}{as^2\times ar^2\times 2F}$ for the fpace defcribed after the rotatory motion ceafes before the motion of the body becomes uniform. Now to determine the fpace defcribed before the rotatory motion was all deftroyed, we have (as the fpace from the end of a uniformly retarded motion varies as the fquare of the velocity) $a^2 : \dfrac{a^2}{2F} :: \dfrac{\overline{a\times ra - v\times rs}^2}{ra^2} : \dfrac{\overline{a\times ra - v\times rs}^2}{2F\times ra^2}$ the fpace that could have been defcribed from the time that the rotatory velocity was deftroyed, until the progreffive motion would have been deftroyed had the fricton continued to act; hence $\dfrac{a^2}{2F} - \dfrac{\overline{a\times ra - v\times rs}^2}{2F\times ra^2} = \dfrac{2av\times ra\times rs - v^2\times rs^2}{2F\times ra^2} =$ the fpace defcribed when the rotatory motion was all deftroyed, hence $\dfrac{\overline{rs^2 + 2rs\times ra\times a\times a - v\times rs}^2}{as^2\times ar^2\times 2F} + \dfrac{2av\times ra\times rs - v^2\times rs^2}{2F\times ra^2} =$ whole fpace defcribed by the body before its motion becomes uniform.

DEFINITION.

The CENTER *of* FRICTION *is that point in the base of a body on which it revolves, into which if the whole surface of the base, and the mass of the body were collected, and made to revolve about the center of the base of the given body, the angular velocity destroyed by its friction would be equal to the angular velocity destroyed in the given body by its friction in the same time.*

PROPOSITION III.

To find the center of friction.

Let FGH (fig. 4.) be the base of a body revolving about its center C, and suppose about *a, b, c,* &c. to be indefinitely small parts of the base, and let A, B, C, &c. be the corresponding parts of the solid, or the prismatic parts having *a, b, c,* &c. for their bases; and P the center of friction. Now it is manifest, that the decrement of the angular velocity must vary as the whole diminution of the momentum of rotation caused by the friction *directly,* and as the whole momentum of rotation or effect of the inertia of all the particles of the solid *inversely;* the *former* being employed in diminishing the angular velocity, and the *latter* in opposing that diminution by the endeavour of the particles to persevere in their motion. Hence, if the effect of the friction varies as the effect of the inertia, the decrements of the angular velocity in a given time will be equal. Now as the quantity of friction (as has been proved from experiments) does not depend on the velocity, the effect of the friction of the elementary parts of the base *a, b, c,* &c.

2   will

will be as $a \times aC$, $b \times bC$, $c \times cC$, &c. alfo the effect of the inertia of the correfponding parts of the body will be as $A \times aC^2$, $B \times bC^2$, $C \times cC^2$, &c. Now when the whole furface of the bafe and mafs of the body are concentrated in P, the effect of the friction will be as $\overline{a + b + c +}$ &c. $\times$ CP, and of the inertia as $\overline{A + B + C +}$ &c. $\times CP^2$; confequently $a \times aC + b \times bC + c \times cC$ $+$ &c. $: \overline{a + b + c +}$ &c. $\times$ CP $:: A \times aC^2 + B \times bC^2 + C \times cC^2$ $+$ &c. $: \overline{A + B + C +}$ &c. $\times CP^2$; and hence

$$CP = \frac{A \times aC^2 + B \times bC^2 + C \times cC^2 + \&c. \times \overline{a + b + c + \&}}{a \times aC + b \times bC + c \times cC + \&c. \times \overline{A + B + C + \&c.}} = \left(\text{if S} = \text{the fum}\right.$$

of the products of each particle into the fquare of its diftance from the axis of motion, $T =$ the fum of the products of each part of the bafe into its diftance from the center, $s =$ the area of the bafe, $t =$ the folid content of the body$\left.\right) \frac{S \times s}{T \times t}$.

PROPOSITION IV.

Given the velocity with which a body begins to revolve about the center of its bafe, to determine the number of revolutions which the body will make before all its motion be deftroyed.

Let the friction, expreffed by the velocity which it is able to deftroy in the body if it were projected in a right line horizontally in one fecond, be determined by experiment, and called F; and fuppofe the initial velocity of the center of friction P about C to be a. Then conceiving the whole furface of the bafe and mafs of the body to be collected into the point P, and (as has been proved in Prop. II.) $\frac{a^2}{2F}$ will be the fpace which the body fo concentrated will defcribe before all its motion be deftroyed;

hence

hence if we put $z = PC$, $p =$ the circumference of a circle whofe radius is unity, then will $pz =$ circumference defcribed by the point P; confequently $\frac{a^2}{2pzF} =$ the number of revolutions required.

Cor. If the folid be a cylinder and r be the radius of its bafe, then $z = \frac{3r}{4}$, and therefore the number of revolutions $= \frac{2a^2}{3prF}$.

PROPOSITION V.

To find the nature of the curve defcribed by any point of a body affeɛted by friɛtion, when it defcends down any inclined plane.

Let efg (fig. 5.) be the body, the points a, r, s, as in Prop. I. and conceive st, rn, to be two indefinitely fmall fpaces defcribed by the points s and r in the fame time, and which therefore will reprefent the velocities of thofe points; but from Prop. I. the ratio of thefe velocities is exprefled by $m \times CB : a \times CA$, hence $st : rn :: m \times CB : a \times CA$. With the center r let a circle vw be defcribed touching the plane LM which is parallel to AC at the point b, and let the radius of this circle be fuch that, conceiving it to defcend upon the plane LM along with the body defcending on CA, the point b may be at reft, or the circle may roll without fliding. To determine which radius, produce rs to x, parallel to which draw ndy, and produce nt to z; now it is manifeft, that in order to anfwer the conditions above-mentioned, the velocity of the point x muft be to the velocity of the point r as $2 : 1$, that is, $zx : yx :: 2 : 1$, hence $zy = yx = nr$. Now $zy : dt (:: ny : nd) :: rx : rs$; therefore $dt = \frac{rs}{rx} \times zy = \frac{rs}{rx} \times nr$, hence $ts (= td + ds = td + nr =$

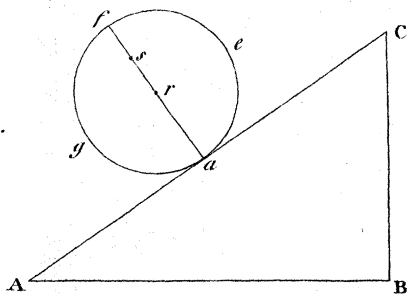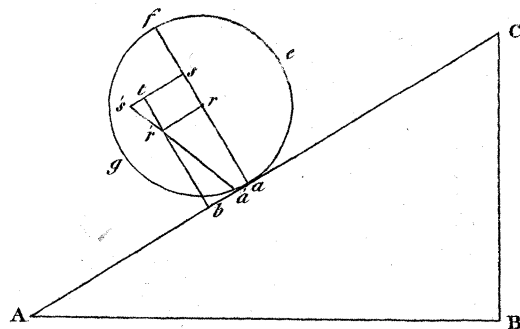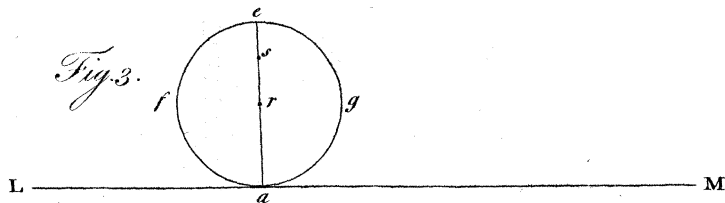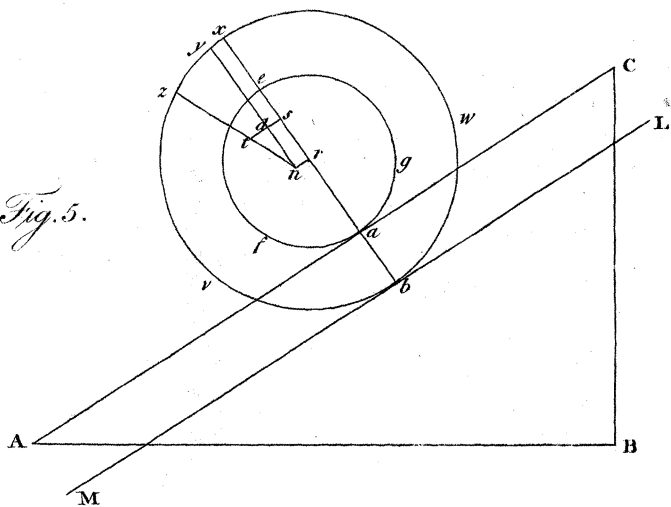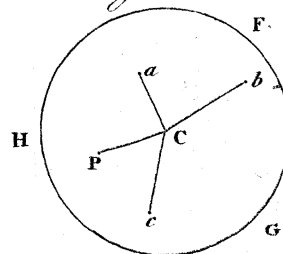4 rs

Fig.1.

Fig.2.

Fig.3.

Fig.4.

Fig.5.

$\frac{rs}{rx} \times nr + nr) = \frac{rs+rx}{rx} \times nr$, confequently $\frac{rs+rx}{rx}$: 1 :: ts : nr ::
(from what is proved above) $m \times$ CB : $a \times$ CA; therefore
$a \times$ CA $\times rs + a \times$ CA $\times rx = m \times$ CB $\times rx$, hence $rx =$
$\frac{a \times \text{CA} \times rs}{m \times \text{CB} - a \times \text{CA}}$ the radius of the circle which rolling down
the inclined plane LM, and carrying the body with it, will
give the true ratio of its progreffive to its rotatory motion,
and confequently that point of the circle which coincides with
any given point of the body will, as the circle revolves upon
the line LM, defcribe the fame curve as the correfponding
point of the body; but as the nature of the curve defcribed by
any point of a circle revolving upon a ftraight line is already
very well known, it feems unneceffary to give the inveftigation.

By a method of reafoning, not very different, may the nature
of the curve, which is defcribed by any point of a body moving
upon an horizontal plane, and affected by friction, be determined.